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TWO CHARACTERISTIC MARKOV-TYPE MAN-POWER FLOW MODELS

Kneale T. Marshall, et al

Naval Postgraduate School
Monterey, California

July 1974

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS-55Mt74071	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Two-Characteristic Markov-Type Manpower Flow Models		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Kneale T. Marshall William J. Hayne		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P.O. 3-0092
11. CONTROLLING OFFICE NAME AND ADDRESS Headquarters, U. S. Marine Corps Washington, D. C. 20380		12. REPORT DATE July 1974
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Manpower
Markov
Personnel planning

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A two-dimensional state space Markov Model of a Manpower System with special structure is analyzed. Examples are given from the military services. The probabilistic properties are discussed in detail with emphasis on computation. The basic equations of manpower stocks and flows are analyzed.

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S/N 0102-014-6601

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I. INTRODUCTION.

The simple fractional flow (or Markov-type) model of personnel movements through an organization has been widely analyzed (see for example Bartholomew (1973), Blumen, Kogan and McCarthy (1955), Lane and Andrew (1955), Rowland and Sovereign (1969) and has been widely applied, especially in military manpower planning (see U.S. Navy (1973)). Other models such as the "cohort" and "chain" models (see Marshall (1973) and Grinold and Marshall (to be published)) satisfy more realistic assumptions on personnel movement, but lack the convenient structure of the Markov model. The purpose of this paper is to present an extension of the Markov Model to one with a 2-dimensional state space. The state space is chosen so that the fractional flow matrix has a special structure which is then exploited.

In section II the structure of the model is presented and in section III examples are given. In section IV we present the probabilistic properties of the model with emphasis on computationally tractable formulae. In section V the structure of the model is exploited in the personnel stock and flow equations.

II. STRUCTURE OF THE FRACTIONAL FLOW MATRIX.

We assume that for planning purposes an organization considers time in discrete periods, and that people are counted at the end of each period. When counted, a person is assumed to possess two characteristics i and j, and is said to be in state (i,j), where i represents the first characteristic (FC), $1 \le i \le n$, and j represents the second characteristic (SC), $\ell(i) \le j \le u(i)$. Here $\ell(i)$ and $u(i)$ are the lower and upper limits respectively for the SC when i is the FC. Also let $J(i) = \{j \mid \ell(i) \le j \le u(i)\}$, the set of SC's for FC i, and let w_i be the number of elements in $J(i)$.

Let $q_i(j,m)$, $j,m \in J(i)$ be the fraction of people in state (i,j) in a time period who move to state (i,m) in the next time period, and let

Q_i be the $w_i \times w_i$ matrix $[q_i(j,m)]$. Let $p_i(j,m)$, $j \in J(i)$, $m \in J(i+1)$ be the fraction of people in state (i,j) in a time period who move to state $(i+1,m)$ in the next time period, and let P_i be the $w_i \times w_{i+1}$ matrix $[p_i(j,m)]$. A basic assumption of our two-characteristic model is that movement in one time period from states with FC i can only be to states with FC i or $i+1$, or out of the system. Following the notation of earlier papers, let Q be the fractional flow matrix for all active states in the system. Then Q has the following structure:

$$Q = \begin{bmatrix} Q_1 & P_1 & & & \\ & Q_2 & P_2 & & \\ & & & & \\ & & & Q_{n-1} & P_{n-1} \\ & & & & Q_n \end{bmatrix} \tag{1}$$

where the zero matrices have been suppressed. Throughout this paper we assume that people can leave the system eventually from any state and thus $(I-Q)$ has an inverse, where I is the identity matrix. This inverse $(I-Q)^{-1}$ is called N, the fundamental matrix (see Kemeny and Snell (1960)). For each i let $A_i = (I-Q_i-P_i)\bar{1}$, where $\bar{1}$ is a vector with every element equal to 1. Then A_i is a vector of w_i elements, each one an attrition fraction from the appropriate state.

III. EXAMPLES.

a) <u>The LOS Model</u>.

Let the "length of service" that a person has completed with an organization be denoted LOS, and let the FC of a person denote his LOS. If the LOS is measured in the same time units as the planning periods then each Q_i matrix is a zero matrix. In each planning period a person's LOS must increase by one unit, so that Q has the structure

$$Q = \begin{bmatrix} 0 & P_1 & & & \\ & 0 & P_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & P_{n-1} \\ & & & & 0 \end{bmatrix} .$$

By appropriate choice of the second characteristic P_i often has special structure too. Consider a hierarchical system where the "rank" or "grade" of an individual is used as his second characteristic. Then the structure of each P_i depends on the organization's promotion scheme. Assume that in one period a person either stays in the same grade or is promoted one grade. No demotions occur. Then P_i would have the structure

$$P_i = \begin{bmatrix} x & x & & & \\ & x & x & & \\ & & \ddots & \ddots & \\ & & & \ddots & x \\ & & & & x \end{bmatrix} ,$$

where x represents a non-zero element.

b) <u>The (Grade, LOS) Model</u>.

If, as in a) no demotions occur, and only single promotions can occur in a time period, then if i indexes the grade, and j the length of service, then Q has the structure shown in (1). Each submatrix has special structure also. Let

q_{ij} = probability a person in state (i,j) at end of one period will be in state $(i,j+1)$ at the end of the next period,

p_{ij} = probability a person in state (i,j) at the end of one period will be in state $(i+1,j+1)$ at the end of the next period.

The transition matrix Q_i has non-zero elements only immediately above the main diagonal:

$$Q_i = \begin{bmatrix} 0 & q_{i,\ell(i)} & & & \\ & 0 & q_{i,\ell(i)+1} & & \\ & & 0 & & \\ & & & & q_{i,u(i)-1} \\ & & & & 0 \end{bmatrix}.$$

The transition matrix P_i has non-zero elements only on a single diagonal band. If $\ell(i+1) \geq \ell(i) + 1$ and $u(i+1) \geq u(i) + 1$, then P_i has the form shown below, where:

1) the top max $\{0,\ell(i+1)-(\ell(i)+1)\}$ rows are zeros,

2) the last max $\{0,u(i+1)-(u(i)+1)\}$ columns are zeros.

$$P_i = \begin{bmatrix} 0 & & & & & \\ 0 & & & & & \\ p_{i,\ell(i+1)-1} & & & & & \\ 0 & p_{i,\ell(i+1)} & & & & \\ & 0 & & & & \\ & & 0 & p_{i,u(i)} & 0 \ldots 0 \end{bmatrix}.$$

If $\ell(i+1) \leq \ell(i)$, the first $\ell(i)+1-\ell(i+1)$ columns of P_i are zeros. If $u(i+1) \leq u(i)$, the bottom $u(i)+1-u(i+1)$ rows of P_i are zeros. Under any circumstances P_i has only one non-zero diagonal, and we call such a matrix a diagonal matrix. Efficient methods of storing and multiplying diagonal matrices are discussed in Hayne and Marshall (1974).

c) The (Grade, TIG) Model.

In certain applications a person's "time in grade," denoted TIG, is more important than his time in the system (LOS). If again we allow no demotions and only single promotions per period and if the FC indexes the grade and the SC the TIG for the appropriate grade, then Q has the same structure (1). Each Q_i has the same structure as in b), but now $\ell(i) = 1$ for each grade i. However, each matrix P_i has a single column of non-zeros,

$$P_i = \begin{bmatrix} p_{i1} & 0 & 0 \\ & & \\ p_{i,u(i)} & 0 & 0 \end{bmatrix},$$

since promotions to the next highest grade lead always to a TIG of 1. Here p_{ij} is the fraction those in grade i, with time in grade i equal to j, who are promoted to grade $i+1$.

If demotions are not allowed and only single promotions can occur per period, then grade is a characteristic which can be used as the FC. A larger number of possibilities occur for the SC. In addition to those above some useful ones are (i) skill category, (ii) physical location in a multi-location organization, and (iii) educational level. Note that educational level could also be used as the FC.

IV. SOME PROBABILISTIC PROPERTIES.

Let T_i be the set of states associate with FC i; thus

$$T_i = \{(i,j) \mid j \in J(i)\}, \qquad i = 1,2,\ldots,n.$$

Also let $w_i = u(i) - \ell(i) + 1$, the number of states in T_i (and $J(i)$). Finally let T_0 be the single state "out of the system."

In this section we develop the probabilistic properties of:

1) any set of states T_i,

2) any union of consequtively indexed sets T_i,

i.e. $\bigcup_{i=k}^{m} T_i$,

3) the union of all transient states, which we call T.

One of the purposes of this development is to show that the stochastic properties of Q, typically a large matrix, are readily calculated in terms of the smaller matrices Q_i and P_i, and as seen in section III these often have extremely simple structure which leads to simple computation.

The format of this section follows closely that of Chapter 3 of Kemeny and Snell (1960). The notation (K&S,3._._) indicates that a result follows from theorem 3._._ in Kemeny and Snell, albeit usually not directly.

a) First-Order Properties.

Recall that we assume system matrix Q has a fundamental matrix $N = (I-Q)^{-1}$, and each element of N is the expected number of visits to the column state starting from the row state (K&S,3.2.4). Since Q has the structure shown in (1),

$$N = \begin{bmatrix} N_1 & N_1P_1N_2 & N_1P_1N_2P_2N_3 & \cdots & \prod_{i=1}^{n-1}(N_iP_i)N_n \\ & N_2 & N_2P_2N_3 & & \prod_{i=2}^{n-1}(N_iP_i)N_n \\ & & N_3 & & \prod_{i=3}^{n-1}(N_iP_i)N_n \\ & & & & \\ & & & & N_n \end{bmatrix}, \tag{2}$$

where $N_i = (I-Q_i)^{-1}$, $i = 1,\ldots,n$, the fundamental matrix for FC i. Note that the large matrix N is completely determined by the matrices N_i and P_i. Thus the only matrix inversions required are those of $(I-Q_i)$, $i = 1,\ldots,n$. This is of considerable computational significance because as previously mentioned Q is usually a large matrix.

Each matrix N_i has a probabilistic interpretation. We pursue this interpretation and show that these matrices can be used to determine other probabilistic properties of interest.

In this section we make numerous definitions and denote the $k^{\underline{th}}$ one by Dk.

Let us consider first the properties associated with a single set of states T_i and define:

D1. $v_i(j,m)$ = expected number of visits to state (i,m) given that FC i is entered in state (i,j),

D2. V_i = a $w_i \times w_i$ matrix having $v_i(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i)+1$.

From (2) the element of N_i in row $j-\ell(i)+1$ and column $m-\ell(i)+1$ equals the expected number of visits to state (i,m) given that FC i is entered in state (i,j). (K&S,3.2.4). So, from definitions D1 and D2, we have,

$$V_i = N_i . \qquad (3)$$

Note that the rows and columns of N_i and V_i correspond to states in T_i in the same order as the rows and columns of Q_i.

Now define:

D3. $\tau_i(j)$ = expected time in FC i given that FC i is entered in state (i,j),

D4. τ_i = $[\tau_i(\ell(i)),\ldots,\tau_i(u(i))]$, a $w_i \times 1$ vector.

The expected time spent in FC i equals the sum of the expected number of visits to the various states in FC i. From (3) and D3,

$$\tau_i(j) = \text{component } (j-\ell(i)+1) \text{ of } N_i\bar{1},$$

and from D4,

$$\tau_i = N_i\bar{1}, \text{ a } w_i \times 1 \text{ vector}, \qquad (4)$$

where $\bar{1}$ is a vector with all components equal to one.

We next turn our attention to where the process goes when it leaves FC i. The process upon leaving T_i must enter either T_{i+1} (if $i < n$) or T_0. Next define

D5. $b_i(j,m)$ = probability of entering FC $i+1$ in state $(i+1,m)$ given that FC i is entered in state (i,j),

D6. B_i = a $w_i \times w_{i+1}$ matrix having $b_i(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i+1)+1$,

D7. $b_i(j)$ = probability of ever entering T_{i+1} given that FC i is entered in state (i,j),

D8. b_i = $[b_i(\ell(i)),\ldots,b_i(u(i))]$, a $w_i \times 1$ vector,

D9. $b_{i0}(j)$ = probability of never entering T_{i+1} given that FC i is entered in state (i,j),

D10. b_{i0} = $[b_{i0}(\ell(i)),\ldots,b_{i0}(u(i))]$, a $w_i \times 1$ vector.

From these definitions it follows that

$$B_i = N_i P_i, \text{ a } w_i \times w_i \text{ matrix (K\&S,3.5.4)}, \tag{5}$$

$$b_i = B_i \bar{1}, \text{ a } w_i \times 1 \text{ vector},$$

and

$$b_{i0} = \bar{1} - b_i$$

$$= N_i A_i, \text{ a } w_i \times 1 \text{ vector}.$$

The matrix B_i is particularly useful in manpower policy analyses. For example let f_i be a $1 \times w_i$ vector of the number of people entering T_i. Then $f_i B_i$ is a $1 \times w_{i+1}$ vector of the number of these people who will eventually enter T_{i+1}. (K&S,3.3.6). Such applications will be the subject of future papers.

Next we consider the first-order properties related to FC's i and k where $i \leq k$. Define:

D11. $b((i,j),(k,m))$ = probability of entering FC k in state (k,m) given that FC i is entered in state (i,j),

D12. B_{ik} = a $w_i \times w_k$ matrix having $b((i,j),(k,m))$ as the element in row $j-\ell(i)+1$ and column $m-\ell(k)+1$.

From definitions D5 and D11 and a simple conditioning argument we have,

$$b((i,j),(i+2,m)) = \sum_{r=\ell(i)}^{u(i)} b_i(j,r) b_{i+1}(r,m).$$

So, from D12,

$$B_{i,i+2} = B_i B_{i+2}.$$

Notice from D11 that B_{ii} is an identity matrix and from D5 that $B_{i,i+1} = B_i$. More generally it can be shown that for $i \leq k$,

$$B_{ik} = \prod_{r=i}^{k-1} B_r, \quad \text{a } w_i \times w_k \text{ matrix.}$$

Define:

D13. $v((i,j),(k,m))$ = expected number of visits to state (k,m) given that FC i is entered in state (i,j),

D14. V_{ik} = a $w_i \times w_k$ matrix having $v((i,j),(k,m))$ as the element in row $j-\ell(i)+1$ and column $m-\ell(k)+1$,

D15. $b_{ik}(j)$ = probability of ever entering FC k, given that FC i is entered in state (i,j),

D16. b_{ik} = $[b_{ik}(\ell(i)),\ldots,b_{ik}(u(i))]$, a $w_i \times 1$ vector.

Considering each row of B_{ik} as the part of an initial probability vector that applies to T_k, we then have,

$$V_{ik} = B_{ik}N_k, \quad \text{a } w_i \times w_k \text{ matrix (K\&S,3.5.4),} \tag{6}$$

and,

$$b_{ik} = B_{ik}\bar{1}, \quad \text{a } w_i \times 1 \text{ vector.}$$

Define:

D17. $\tau_{ik}(j)$ = expected time in FC k given that FC i was entered in state (i,j),

D18. τ_{ik} = $[\tau_{ik}(\ell(i)),\ldots,\tau_{ik}(u(i))]$, a $w_k \times 1$ vector.

The expected time in an FC is the sum of the expected number of visits to states in that FC, so

$$\tau_{ik} = V_{ik}\bar{1}, \quad \text{a } w_i \times 1 \text{ vector.}$$

This completes our study of the first-order properties related to the various FC's of the system. The foregoing definitions by no means exhaust the first-order properties of the two-characteristic model that might conceivably be of interest. It is felt, however, that these properties will often be of practical interest and that other first-order properties may be readily derived from those given above.

b) Two Special Cases.

The elements of the fundamental matrix for FC i, N_i have a somewhat different interpretation when the states in FC i have what we call the "0-1 visiting property." We say that a state has the 0-1 visiting property if the state can be visited no more than one time. Important examples of two-characteristic models in which all transient states have the 0-1 visiting property are the models in which the FC or SC is length of service or where the SC is time in grade.

If each state in T_i has the 0-1 visiting property, then the expected number of visits to a state in T_i is equal to the probability of visiting the state. The element of N_i in row $j-\ell(i)+1$ and column $m-\ell(i)+1$ may then be interpreted as the probability of visiting state (i,m) given that FC i is entered in state (i,j).

Another property of interest is the "no return property." We say that a set of states has the no return property if it is impossible to ever make a transition into the state after a transition has been made out of the state. The 0-1 visiting property implies the no return property, but they are not equivalent. For example, in modelling manpower flows in the U.S. Civil Service one might use "GS grade" as the FC and "pay step" as the SC. Each state is then a couple (grade, pay step). A person can stay in the same pay step for

more than one period, so if there are no demotions then each state would have the no return property but not the 0-1 visiting property.

If the states in T_i have the no return property then it is possible to order the states in T_i so that Q_i is upper triangular. When Q_i is upper triangular so is $I-Q_i$ and the computation of the inverse of $I-Q_i$, i.e. the fundamental matrix for FC i, N_i, is considerably easier than in the general case.

If the states in T_i have the 0-1 visiting property, then not only is N_i upper triangular but also the elements of N_i on the main diagonal are all ones.

c) Variances.

The format in this section follows closely that of section a), but here we are concerned with various second moment properties of the two-characteristic model.

Define:

D19. $v_{2,i}(j,m)$ = variance of the number of visits to state (i,m) given that FC i is entered in state (i,j),

D20. $V_{2,i}$ = a $w_i \times w_i$ matrix having $v_{2,i}(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i)+1$.

Following (K&S,3.3.3),

$$V_{2,i} = N_i(2(N_i)_{dg}-I) - (N_i)_{sq},$$

where for any square matrix A, A_{dg} and A_{sq} both have the same dimensions as A; A_{dg} is defined when A is square and is formed by setting all elements in A not on the main diagonal to zero; A_{sq} is formed by squaring all the elements in A.

Define:

D21. $\tau_{2,i}(j)$ = variance of the time spent in FC i given that FC i is entered in state (i,j)

D22. $\tau_{2,i} = [\tau_{2,i}(\ell(i)),\ldots,\tau_{2,i}(u(i))]$, a $w_i \times 1$ vector.

Following (K&S,3.3.5),

$$\tau_{2,i} = (2N_i-I)\tau_i - (\tau_i)_{sq}.$$

Define:

D23. $v_2((i,j),(k,m))$ = variance of the number of visits to state (k,m) given that FC i is entered in state (i,j),

D24. $V_2(i,k)$ = a $w_i \times w_k$ matrix having $v_2((i,j),(k,m))$ as the element in row $j-\ell(i)+1$ and column $m-\ell(k)+1$,

Following (K&S,3.3.6),

$$V_2(i,k) = V_{ik}(2(N_k)_{dg}-I) - (V_{ik})_{sq}.$$

Define:

D26. $\tau_2((i,j),k)$ = variance of time spent in FC k given that FC i is entered in state (i,j),

D27. $\tau_2(i,k) = [\tau_2((i,\ell(i)),k),\ldots,\tau_2((i,u(i)),k)]$, a $w_i \times 1$ vector.

Following (K&S,3.3.6),

$$\tau_2(i,k) = B_{ik}(2N_k-I)\tau_k - (\tau_{ik})_{sq}.$$

If each state in T_i has the 0-1 visiting property, then the diagonal elements of N_i are equal to one, and,

$$(N_i)_{dg} = I,$$

$$V_{2,i} = N_i - (N_i)_{sq},$$

$$V_2(i,k) = V_{ik} - (V_{ik})_{sq}.$$

d) <u>Matrices of t-Step Transition Probabilities</u>.

In this section we consider the probability of being in state (k,m) t steps after being in state (i,j). The matrices of these probabilities are called the t-step transition matrices. They are used in section V to represent the stock vectors as a sum of steady-state and transient components.

Define:

D28. $m(t:(i,j),(k,m))$ = probability of being in state (k,m) t steps after being in state (i,j), $t = 0,1,2,\ldots$

D29. $M_{ik}(t)$ = a $w_i \times w_k$ matrix having $m(t;(i,j),(k,m))$ as the element in row $j-\ell(i)+1$ and column $m-\ell(k)+1$.

The rows of $M_{ik}(t)$ are associated with states in T_i; the columns of $M_{ik}(t)$ are associated with states in T_k.

We have immediately that

$$M_{ii}(0) = I.$$

From our assumptions on the structure of Q we have,

$$M_{ik}(t) = 0 \quad \text{if} \quad i > k,$$

$$M_{ik}(t) = 0 \quad \text{if} \quad t < k-i.$$

If the process is to be in state (k,m) exactly t steps after being in state (i,j), then it must be in some state with FC k or $k-1$ exactly $t-1$ steps after being in state (i,j). Conditioning on this fact leads to the recursive equation,

$$M_{ik}(t) = M_{ik}(t-1)Q_i + M_{i,k-1}(t-1)P_{i-1}, \qquad t = 1,2,\ldots \tag{7}$$

For any i and k the sum over t of the probability matrices $M_{ik}(t)$ gives the matrix of the expected number of visits to states with FC k starting from states with FC i. So we have,

$$\begin{aligned} \sum_{t=0}^{\infty} M_{ik}(t) &= V_{ik}, \qquad i \le k, \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{8}$$

Recall that the Q_i matrices are transient, so V_{ik} is matrix of finite elements. This implies that,

$$\lim_{t\to\infty} M_{ik}(t) = 0. \tag{9}$$

From (7) it can be shown by an inductive argument that

$$M_{ik}(t) = \sum_{r=0}^{t-1} M_{i,k-1}(t-1-r)P_{k-1}Q_k^r. \tag{10}$$

The t-step transition matrices provide a rather comprehensive picture of how people move through a two-characteristic system.

e) <u>Conditioning on Promotion When the FC is Grade</u>.

In manpower planning one is often interested in conditional probabilities, e.g. the probability of attaining grade k given that grade i is attained. The stochastic properties of the transient matrix Q under conditioning on promotion when the FC is grade are briefly developed in this section.

Define:

D30. $(i,j;t)$ = the event "in state (i,j) at time t"

D31. T_k^* = the event "a transition is made into T_k before leaving the system."

Conditioning on the event T_k^* is the same as conditioning on promotion to grade k.

Define:

D32. $q_i(j,m) = \Pr[(i,m;t+1)|(i,j;t)]$

D33. $q_i^*(j,m) = \Pr[(i,m;t+1)|(i,j,t),T_{i+1}^*]$

Provided that $\Pr[T_{i+1}^*|(i,j;t)] \neq 0$, we have by conditional arguments,

$$q_i^*(j,m) = q_i(j,m) \times \frac{b_i(m)}{b_i(j)} . \tag{11}$$

Define:

D34. C_i = a $w_i \times w_i$ matrix having the elements of b_i (see D8) on its main diagonal and zeros elsewhere.

We assume that promotion to grade $i+1$ is possible from every state in T_i. Under this assumption C_i^{-1} exists. If promotion to grade $i+1$ is impossible from some state (i,j) then we must avoid conditioning on an impossible event. This is readily accomplished by temporarily treating state (i,j) as part of T_0 (out of the system) and redefining $J(i)$, Q_i, P_i and A_i accordingly.

Define:

D35. Q_i^* = a $w_i \times w_i$ matrix having $q_i^*(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i)+1$.

Then from (11) and D34,

$$Q_i^* = C_i^{-1}Q_iC_i .$$

The matrix Q_i^* is the matrix of within grade one-step transition probabilities conditioned on the attainment of grade $i+1$.

Define:

D36. $p_i(j,m) = \Pr[(i+1,m;t+1) \mid (i,j;t)]$

D37. $p_i^*(j,m) = \Pr[(i+1,m;t+1) \mid (i,j;t), T_{i+1}^*]$

D38. P_i^* = a $w_i \times w_{i+1}$ matrix having $p_i^*(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i+1)+1$.

We then have,

$$p_i^*(j,m) = p_i(j,m) \times \frac{1}{b_i(j)} .$$

Thus from D34 and D38,

$$P_i^* = C_i^{-1} P_i .$$

The matrix P_i^* is the matrix of one-step promotion probabilities conditioned on the attainment of grade $i+1$.

Because $(Q_i^*)^r = C_i^{-1} Q_i^r C_i$,

the fundamental matrix for grade i when we condition on promotion to grade $i+1$ is

$$\begin{aligned} N_i^* &= (I-Q_i^*)^{-1} \\ &= \sum_{r=0}^{\infty} (Q_i^*)^r \\ &= \sum_{r=0}^{\infty} C_i^{-1} Q_i^r C_i \\ &= C_i^{-1} N_i C_i . \end{aligned}$$

Define:

D39. $v_i^*(j,m)$ = expected number of visits to state (i,m) given that grade i is entered in state (i,j) and grade $i+1$ is attained.

D40. V_i^* = a $w_i \times w_i$ matrix having $v_i^*(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i)+1$

D41. $b_i^*(j,m)$ = probability of entering grade $(i+1)$ in state $(i+1,m)$ given that grade i is entered in state (i,j) and grade $i+1$ is attained.

D42. B_i^* = a $w_i \times w_{i+1}$ matrix having $b_i^*(j,m)$ as the element in row $j-\ell(i)+1$ and column $m-\ell(i+1)-1$.

Then one may show that,

$$V_i^* = N_i^*,$$

and

$$B_i^* = N_i^* P_i^*$$
$$= C_i^{-1} B_i .$$

Note that B_i^* is simply B_i with its rows normalized, but Q_i^* is <u>not</u> simply a row normalized form of Q_i.

As with the matrices B_i, products of matrices B_i^* with successive indices are well defined: their meaning is that of a matrix B_{ik} as defined in D11 and D12 with conditioning on attainment of grade k.

The conditioned and unconditioned matrices may be used together. For example, the elements of $B_i^* B_{i+1}$ give the probabilities of entering grade $i+2$ in the column state conditioned on starting from the row state in T_i and eventually attaining grade $i+2$.

V. EQUATIONS OF STOCKS AND FLOWS.

We begin by defining the terms "stocks" and "flows," and then discuss why stocks and flows are important in manpower planning models Next the

relations between stocks and flows in a two-characteristic model are developed. Finally, we show how the stocks can be represented as the sum of a "steady-state" component and a "transient" component.

a) Definitions and Background.

A period is the interval of time from immediately after an integer value of the time parameter t up to and including the next integer value of t. A period is identified by the value of the time parameter at the end of the period. Thus,

$$\text{period } t_1 = \{t: \ t_1 - 1 < t \leq t_1\}$$

where t_1 is an integer.

The number of people in a state at the end of a period is referred to as the "stock" in that state. Thus, stocks are counted only at integer values of the time parameter t.

The number of people who change their status in the system from one state to another during any period is referred to as a "flow." Flows occur during a period, but we do not specify the exact time at which they occur.

Stocks and flows are of primary importance in most manpower planning models. The most obvious reason for this is that costs are closely related to stocks and flows, e.g. total payroll depends on stocks, transportation costs or retraining costs depend on flows. Recruiting policy and promotion policy depend in the short term on present stocks and in the long term on how we model future stocks and flows. Determining the feasibility of a retirement plan and evaluating the effects of a change in billet structure are other instances in which the planner needs to be able to model stocks and flows in a manpower system.

We now define the variables that are used to model the stocks and flows in the two-characteristic model. Recall that T_i is the set of states associated with FC i, w_i is the number of states in T_i, and for convenience of notation we assume the second characteristic takes on successive integer values for FC i.

In a Markov model the stocks and flows are in general random variables. In this section we deal only with the expected values of stocks and flows. Such a model is called a "fractional flow model" because the transition probabilities of the Markov model are in effect treated as fractions which direct flows through the system in a deterministic manner.

Let,

$s_{ij}(t)$ = expected stocks in state (i,j) at time t,

$s_i(t) = (s_{i,\ell(i)}(t),\ldots,s_{i,u(i)}(t))$,
a $1 \times w_i$ vector of expected stocks in T_i,

$s(t) = (s_1(t),\ldots,s_n(t))$,
a $1 \times \sum_{i=1}^{n} w_i$ vector of expected stocks in the system.

By our basic assumption, flows into any state in T_i must come from a state in either T_i or T_{i-1}. We also make provision in our model for "external flows." The source of such flows is unspecified. However, we may consider external flows as consisting of people hired into the system. The external flows may be deterministic or random, but we deal only with their expected values.

Let,

$d_{ij}(t)$ = expected flow from states in T_i to state (i,j) during period t, a scalar;

$d_i(t) = (d_{i,\ell(i)}(t),\ldots,d_{i,u(i)}(t))$, a $1 \times w_i$ vector;

$f_{ij}(t)$ = expected external flow into state (i,j) during period t, a scalar;

$f_i(t) = (f_{i,\ell(i)}(t),\ldots,f_{i,u(i)}(t))$, a $1 \times w_i$ vector;

$g_{ij}(t)$ = expected flow from states in T_{i-1} to state (i,j) during period t, a scalar;

$g_i(t) = (g_{i,\ell(i)}(t),\ldots,g_{i,u(i)}(t))$, a $1 \times w_i$ vector.

When $i = 1$, $g_{ij}(t)$ is defined to be zero.

The relation between the flow vectors and the stock vector in grade i is depicted in Figure 1 where "$T_i;t$" denotes the states with FC i at time t.

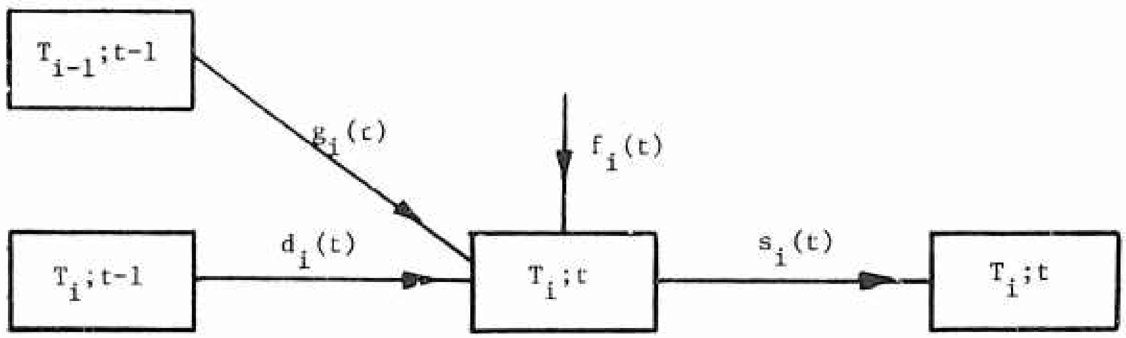

Figure 1. Stocks and Flow with FC i in Period t.

b) Basic Stock Equation.

Clearly, from our assumptions,

$$s_i(t) = d_i(t) + f_i(t) + g_i(t).$$

(See Figure 1.)

It will be convenient to define,

$$s_0(t) = 0, \quad \text{a vector of zeros,}$$
$$P_0 = 0, \quad \text{a matrix of zeros.}$$

Using conditional expectation we then have

$$d_i(t) = s_i(t-1)Q_i, \qquad i = 1,\ldots,n,$$
$$g_i(t) = s_{i-1}(t-1)P_{i-1}, \qquad i = 1,\ldots,n.$$

The basic stock equation is then,

$$s_i(t) = s_i(t-1)Q_i + f_i(t) + s_{i-1}(t-1)P_{i-1}, \qquad i = 1,\ldots,n. \tag{12}$$

The basic stock equation for FC i can be written in terms of the expected or actual stocks with FC i in previous periods. By recursively applying the basic stock equation for $s_i(t), s_i(t-1), \ldots, s_i(1)$ one obtains

$$s_i(t) = s_i(0)Q_i^t + \sum_{r=0}^{t-1} f_i(t-r)Q_i^r + \sum_{r=0}^{t-1} s_{i-1}(t-r-1)P_{i-1}Q_i^r,$$
$$t = 0,1,2,\ldots, \tag{13}$$
$$i = 1,\ldots,n,$$

which we will refer to as the cumulative stock equation.

Equations (12) and (13) are used frequently in the remainder of this paper. Some manpower models used in the U.S. military for short-range forecasting consist principally of an application of an equation similar to (12).

c) Transient Properties of the Stocks.

In this section we develop a method for expressing the stock vector as a sum of a "steady-state" component and a "transient" component. This method helps one to understand how the stock vectors change in going from any present stock vector to future stock vectors. This method also helps one interpret the character of the limiting stock vector.

We do not want to restrict ourselves to cases in which the stock vector converges (as t increases) to a finite vector. We say that the vector function $\tilde{s}_i(t)$ is a steady-state component of the stock vector $s_i(t)$ if,

$$\lim_{t\to\infty} (s_i(t)-\tilde{s}_i(t)) = 0.$$

For any sequence of stock vectors $\langle s_i(t)\rangle$ there is more than one choice of the steady-state component. In applications one would prefer a steady state component having a relatively simple mathematical form. We show that in some cases a judicious choice of $\tilde{s}_i(0)$ makes this possible. The following theorem shows the properties of a class of steady-state components which are quite useful.

Theorem. For any collection of $1 \times w_i$ vectors $\tilde{s}_i(0)$, $i = 1,\ldots,n$, let the vector functions $\tilde{s}_i(t)$ satisfy

$$\tilde{s}_i(t) = \tilde{s}_i(t-1)Q_i + f_i(t) + \tilde{s}_{i-1}(t-1)P_{i-1}, \qquad t = 1,2,\ldots, \quad i = 1,\ldots,n,$$

i.e. the vector functions $\tilde{s}_i(t)$ satisfy the basic stock equation (12). Then

(a) the actual stocks at time t are

$$s_i(t) = \tilde{s}_i(t) + \sum_{k=1}^{i} (s_k(0)-\tilde{s}_k(0))M_{ki}(t),$$

(b) $$\sum_{t=0}^{\infty} (s_i(t)-\tilde{s}_i(t)) = \sum_{k=1}^{i} (s_k(0)-\breve{s}_k(0))B_{ki}N_i,$$

a $1 \times w_i$ vector having finite components,

(c) $\tilde{s}_i(t)$ is a steady-state component of the stock vector $s_i(t)$, i.e.

$$\lim_{t\to\infty} (s_i(t)-\tilde{s}_i(t)) = 0.$$

Before proving the theorem we explain why one might be interested in such a theorem. Part (c) of the theorem says that $\tilde{s}_i(t)$ is a steady-state component of the stock vector $s_i(t)$, and part (a) shows how the stock vector $s_i(t)$ can be expressed as the sum of a steady-state component and a transient component. Part (b) of the theorem says that the total over all periods of the difference between the stock vector and its steady-state component is a readily calculated finite vector.

Such information can be useful when long-range planning has been done using an "equilibrium model." As an example consider an organization which intends to change from its present size of 250,000 to a size of 200,000. The manpower planner may use an equilibrium model to develop policies that are in some sense optimal, and these policies will maintain the size of the organization at 200,000 people once the organization has been reduced to this size. So the equilibrium model tells the planner what to do once the size of the organization reaches the desired equilibrium level but it doesn't tell him how to change the organization from its present level (250,000) to the desired equilibrium level (200,000). This problem of finding an optimal transition policy to go from present stock levels to a future equilibrium stock distribution is a very difficult one (see Chapter 4 , Bartholomew (1973)). One method for making the transition is to immediately implement

the hiring, promotion and attrition policies that have been derived from the equilibrium model. Because of the transient nature of the system these policies will eventually bring the stocks in the system to their equilibrium levels.

In the theorem the vector functions $\tilde{s}_i(t)$ play the role of what the stocks would be at time t if the system were in equilibrium. The stock vectors $s_i(t)$ indicate what the stocks will be at time t if we start with the present stocks $s_i(0)$ and implement the policies of the equilibrium model (which are reflected in the external flows, $f_i(t)$, and the transition matrices Q_i, P_i and A_i). From part (a) of the theorem we may readily calculate the difference between actual stocks and equilibrium stocks in any grade and any period. If there is a penalty associated with having more people than the equilibrium stocks in the system, then part (b) of the theorem may be used to calculate the total penalty. Part (c) of the theorem assures the planner that the difference between the actual and equilibrium stocks does converge to a zero vector as the time parameter t increases.

The proof of the theorem follows.

Proof. By hypothesis the vector functions $\tilde{s}_i(t)$ satisfy the basic stock equation (12), so they must also satisfy the cumulative stock equation (13):

$$\tilde{s}_i(t) = \tilde{s}_i(0)Q_i^t + \sum_{r=0}^{t-1} f_i(t-r)Q_i^r + \sum_{r=0}^{t-1} s_{i-1}(t-r-1)P_{i-1}Q_i^r.$$

Of course the stock vectors $s_i(t)$ also satisfy the cumulative stock equation (13), so we have,

$$s_i(t) - \tilde{s}_i(t) = (s_i(0)-\tilde{s}_i(0))Q_i^t + \sum_{r=0}^{t-1} (s_{i-1}(t-r-1)-\tilde{s}_{i-1}(t-r-1))P_{i-1}Q_i^r.$$

When $i = 1$ this implies,

$$s_1(t) = \tilde{s}_1(t) + (s_1(0)-\tilde{s}_1(0))Q_1^t$$

$$= \tilde{s}_1(t) + \sum_{k=1}^{1} (s_k(0)-\tilde{s}_k(0))M_{k,1}(t),$$

so we have shown that part (a) of the theorem is true when $i = 1$. Suppose part (a) of the theorem is true for grade $i-1$, i.e.

$$s_{i-1}(t) = \tilde{s}_{i-1}(t) + \sum_{k=1}^{i-1} (\tilde{s}_k(0)-s_k(0))M_{k,i-1}(t).$$

Then,

$$s_{i-1}(t-r-1)-\tilde{s}_{i-1}(t-r-1) = \sum_{k=1}^{i-1} (s_k(0)-\tilde{s}_k(0))M_{k,i-1}(t-r-1),$$

and

$$s_i(t) - \tilde{s}_i(t) = (s_i(0)-\tilde{s}_i(0))Q_i^t + \sum_{r=0}^{t-1}\sum_{k=1}^{i-1} (s_k(0)-\tilde{s}_k(0))M_{k,i-1}(t-r-1)P_{i-1}Q_i^r$$

$$= (s_i(0)-\tilde{s}_i(0))Q_i^t + \sum_{k=1}^{i-1} (s_k(0)-\tilde{s}_k(0)) \sum_{r=0}^{t-1} M_{k,i-1}(t-r-1)P_{i-1}Q_i^r$$

From equation (10) in Section IV

$$\sum_{r=0}^{t-1} M_{k,i-1}(t-r-1)P_{i-1}Q_i^r = M_{ki}(t),$$

so we have shown by induction that,

$$s_i(t) - \tilde{s}_i(t) = (s_i(0)-\tilde{s}_i(0))Q_i^t + \sum_{k=1}^{i-1} (s_k(0)-\tilde{s}_k(0))M_{ki}(t).$$

This proves part (a) of the theorem.

From part (a),

$$\sum_{t=0}^{\infty} (s_i(t)-\tilde{s}_i(t)) = \sum_{t=0}^{\infty}\sum_{k=1}^{i} (s_k(0)-\tilde{s}_k(0))M_{ki}(t)$$

$$= \sum_{k=1}^{i} (s_k(0)-\tilde{s}_k(0)) \sum_{t=0}^{\infty} M_{ki}(t)$$

$$= \sum_{k=1}^{i} (s_k(0)-\tilde{s}_k(0))B_{ki}N_i,$$

a $1 \times w_i$ vector having finite components.

The last step above follows from equations (6) and (8) of Section IV. This proves part (b) of the theorem.

Part (c) follows from the fact that the sum in part (b) is finite, and the proof of the theorem is complete.

The utility of this approach depends on our ability to find vectors $\tilde{s}_i(0)$ such that the vector functions $\tilde{s}_i(t)$ are simple and readily calculated. Some examples follow.

1. Fixed External Flows.

The equilibrium models previously mentioned enjoy some popularity in military manpower planning in the United States (see for example, RAND Corporation (1973)). The rationale underlying the use of such models is that one should determine the organization structure and the policies to maintain this structure which are optimal (or "least infeasible"). Among the policies derived from an equilibrium model is the hiring policy. This has the form,

$$f_i(t) = f_i, \qquad t = 1,2,\ldots,$$

$$i = 1,\ldots,n$$

where the vector of the number of people to be hired into the states in grade i each period, f_i, is specified from the equilibrium model.

Define,

$$\tilde{s}_1(0) = f_1N_1.$$

Then using (12) it is easy to show that

$$\tilde{s}_1(t) = f_1N_1 \quad \text{for all } t.$$

Thus, from the theorem

$$s_1(t) = \tilde{s}_1(t) + (s_1(0)-\tilde{s}_1(0))M_{11}(t)$$

$$= f_1N_1 + (s_1(0)-f_1N_1)Q_1^t.$$

Now recursively define,

$$\tilde{s}_1 = \tilde{s}_1(t) = f_1N_1,$$

$$\tilde{s}_i = (f_i+\tilde{s}_{i-1}P_{i-1})N_i, \qquad i = 2,\ldots,n. \tag{14}$$

It is straightforward to verify that these $\tilde{s}_i$ satisfy the basic stock equation (12), so we have from the theorem, when $f_i(t) = f_i$,

$$s_i(t) = \tilde{s}_i + \sum_{k=1}^{i} (s_k(0)-\tilde{s}_k)M_{ki}(t).$$

The steady-state component can also be written,

$$\tilde{s}_i = \sum_{k=1}^{i} f_kB_{ki}N_i, \qquad i = 1,\ldots,n. \tag{15}$$

Note that $\sum_{k=1}^{i} f_kB_{ki}$ is a non-negative $1 \times w_i$ vector, so the limiting vector of stocks in grade i must be a non-negative combination of the rows of N_i. Thus, in general, not all non-negative $1 \times w_i$ vectors are possible limiting stock vectors under constant external flows.

2. Linear Growth of External Flows.

In this section we consider the case in which the number of people hired into each state increases by the same amount each period. Such a hiring policy may not be natural over a long period of time, but it may provide a simple approximation to planned hiring policies.

Let the $1 \times w_i$ vector f_i be the increase in the number hired into states with FC i each period. Then the external flow vector for FC i is,

$$f_i(t) = tf_i, \qquad t = 1,2,\ldots,$$
$$i = 1,\ldots,n.$$

Let,

$$\tilde{s}_1(0) = -f_1N_1Q_1N_1.$$

Let the vector function $\tilde{s}_1(t)$ satisfy the basic stock equation (1),

$$\tilde{s}_1(t) = \tilde{s}_1(t-1)Q_1 + f_1(t).$$

Using the identity $N_1Q_1+I = N_1$ one can show that

$$\tilde{s}_1(t) = tf_1N_1 - N_1Q_1N_1.$$

Thus from the theorem,

$$s_1(t) = tf_1N_1 - f_1N_1Q_1N_1 + (s_1(0)+f_1N_1Q_1N_1)Q_1^t.$$

We note that $s_1(t)$ is of the form

$$\tilde{s}_1(t) = tL_1 + C_1$$

where $L_1 = f_1N_1$ is a $1 \times w_i$ vector,

and $C_1 = -f_1N_1Q_1N_1$ is a $1 \times w_i$ vector.

Consider some FC $i \in \{2,\ldots,n\}$. Suppose that

$$\tilde{s}_{i-1}(t) = tL_{i-1} + C_{i-1},$$

where L_{i-1} and C_{i-1} are $1 \times w_{i-1}$ vectors.

Using the identity

$$(tf_iN_i-f_iN_iQ_iN_i)Q_i + (t+1)f_i = ((t+1)f_iN_i-f_iN_iQ_iN_i),$$

one may show that if

$$\tilde{s}_i(t) = tf_iN_i - f_iN_iQ_iN_i + \tilde{s}_{i-1}(t-1)P_{i-1}N_i - L_{i-1}P_{i-1}N_iQ_iN_i,$$

then $\tilde{s}_i(t)$ satisfies the basic stock equation (12). Note that $\tilde{s}_i(t)$ has the form,

$$\tilde{s}_i(t) = tL_i + C_i,$$

where,

$$\begin{aligned} L_i &= f_iN_i + L_{i-1}P_{i-1}N_i \\ &= (f_i+L_{i-1}P_{i-1})N_i, \end{aligned} \qquad (16)$$

and,

$$\begin{aligned} C_i &= -(f_i+L_{i-1}P_{i-1})N_iQ_iN_i - (L_{i-1}-C_{i-1})P_{i-1}N_i \\ &= -((L_{i-1}-C_{i-1})P_{i-1} + f_iN_iQ_i)N_i. \end{aligned}$$

Thus we have shown that when the external flows grow linearly the steady-state component of the stocks also grows linearly.

By recursive substitution in (16) we have,

$$L_i = \sum_{k=1}^{i} f_kB_{ki}N_i.$$

Note that this vector gives the expected number of visits to states with FC i of $f_k = f_k(t+1) - f_k(t)$ entrants with FC k, $k = 1,\ldots,i$. That is, the growth in the stocks with FC i each period, L_i, equals the expected number of visits to FC i of the growth in the external flows each period in the FC's less than or equal to i.

Both L_i and C_i have the fundamental matrix N_i as a right factor, so the steady state component of the stock vector, $s_i(t)$ must be a non-negative combination of the rows of N_i. This same result was observed in the case of constant external flows.

In summary we have shown that by choosing

$$\tilde{s}_i(t) = tL_i + C_i$$

where

$$L_i = f_1 N_1 \quad \text{when} \quad i = 1,$$

$$= (f_i + L_{i-1} P_{i-1}) N_i, \qquad i = 2,\ldots,n,$$

and

$$C_i = -f_1 N_1 Q_1 N_1 \quad \text{when} \quad i = 1,$$

$$= -((L_{i-1} - C_{i-1}) P_{i-1} + f_i N_i Q_i) N_i, \qquad i = 2,\ldots,n,$$

then from the theorem the stock equation may be written

$$s_i(t) = \tilde{s}_i(t) + \sum_{k=1}^{i} (s_k(0) - \tilde{s}_k(0)) M_{ki}(t).$$

3. Geometric Growth of External Flows.

In this subsection we show that geometric growth of external flows leads (eventually) to geometric growth of the stocks. We consider the case in which the external flows into the states in grade i are proportional to a known vector f_i and grow geometrically at a rate θ_i. Thus,

$$f_i(t) = \theta_i^t f_i, \qquad t = 1,2,\ldots,$$
$$i = 1,\ldots,n$$
$$\theta_i > 0.$$

When $0 < \theta_i < 1$, the external flows contract rather than grow.

If θ_k is not an eigenvalue of Q_i for $k \leq i \leq n$ we may define,

$$N_i(\theta_k) = (I - \frac{1}{\theta_k} Q_i)^{-1}.$$

If the states in grade i have the 0-1 visiting property then all eigenvalues of Q_i are zero and thus $\theta_k > 0$ is never equal to an eigenvalue of Q_i in this case.

The following identity will be useful:

$$N_i(\theta_k) = \sum_{r=0}^{\infty} (\frac{1}{\theta_k} Q_i)^r$$

$$N_i(\theta_k)Q_i = \theta_k \sum_{r=0}^{\infty} (\frac{1}{\theta_k} Q_i)^{r+1}$$

$$= \theta_k(-I+N_i(\theta_k)).$$

Define,

$$\tilde{s}_1(0) = f_1 N_1(\theta_1).$$

Then it can be shown that if

$$\tilde{s}_1(t) = \theta_1^t f_1 N_1(\theta_1),$$

then $\tilde{s}_1(t)$, $t = 0,1,\ldots,$ satisfies the basic stock equation, and from the theorem,

$$s_1(t) = \theta_1^t f_1 N_1(\theta_1) + (s_1(0)-f_1 N_1(\theta_1))M_{11}(t).$$

Note that the steady-state component of the FC 1 stock vector grows geometrically at the same rate as the external flows into FC 1.

Define,

$$B_{ki}(\theta_k) = \prod_{m=k}^{i-1} (N_m(\theta_k)P_m), \qquad 1 \leq k \leq i \leq n.$$

Then it can be shown that if

$$\tilde{s}_i(t) = \sum_{k=1}^{i} \theta_k^{t-(i-k)} f_k B_{ki}(\theta_k) N_i(\theta_k)$$

then $\tilde{s}_i(t)$, $t = 0,1,\ldots,$ satisfies the basic stock equation (12). Note that in the limit the stocks with FC i grow geometrically at the rate of the largest θ_k where $k \leq i$.

Define,

$$\theta_M = \max\{\theta_k;\ k=1,\ldots,i\},$$

The steady-state component of the stock vector is not in general a non-negative combination of the rows of N_i (as was the case with constant external flows and linear growth of external flows). Rather the steady-state stock distribution is a non-negative combination of the rows of $N_i(\theta_M)$. The rows of $N_i(\theta_M)$ need not be non-negative combinations of the rows of N_i, so the limiting stock distributions that are possible under geometric growth of external flows need not be the same as the limiting stock distributions under constant external flows and linear growth of external flows.

REFERENCES

Bartholomew, D. J., Stochastic Models for Social Processes, 2nd Edition, Wiley, 1973.

Blumen, I., Kogan, M. and McCarthy, P., The Industrial Mobility of Labor as a Probability Process. Cornell University, 1955.

Grinold, R. C. and Marshall, K. T., Manpower Planning Models, to be published.

Kemeny, J. G. and Snell, J. L., Finite Markov Chains, Van Nostrand, 1960.

Lane, K. F. and Andrew, J. E., "A Method of Labour Turnover Analysis," J. R. Stat. Soc., Vol. A118, pp. 296-323, 1955.

Marshall, K. T., "A Comparison of Two Personnel Prediction Models," Operations Research, Vol. 21, No. 3, pp. 810-822, 1973.

RAND Corporation, "Planning in Large Personnel Systems: A Reexamination of the TOPLINE Static Planning Model," R-1274-PR, 1973.

Rowland, K. M. and Sovereign, M. G., "Markov-Chain Analysis of Internal Manpower Supply," Industrial Relations, Vol. 9, No. 1, pp. 88-99, 1969.

U. S. Navy, "Computer Models for Manpower and Personnel Management: State of Current Technology," (NAMPS Project Report 73-2), Naval Personnel Research and Development Laboratory, 1973.